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RESPONSE OF CIRCULAR PLATES TO CENTRAL PULSE LOADING

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Abstract—An analysis is presented for the response of a clamped circular plate subjected to a rectangular pulse uniformly distributed over a central circular area. The plate is rigid-perfectly plastic with yielding according to the Johansen criterion and the associated flow rule. Bending is assumed to be the predominant response. Simple formulas are obtained for the permanent central deflection for all pressures and loaded areas.

NOTATION

a	radius of loaded area
m	areal mass of plate
p	pressure
p_d	rectangular pulse pressure
p_s	plate static collapse pressure
r	radial coordinate
r_0	radius of inner hinge circle
$\bar{r}_0$	initial radius of inner hinge circle
r_1	radius of outer hinge circle
$\bar{r}_1$	initial radius of outer hinge circle
t	time
t_a	time when outer hinge reaches support before $r_1 = 0$.
t_b	time when inner hinge reaches center before $r_0 = R$
t_{ab}	time when inner hinge reaches center after $r_0 = R$
t_{ba}	time when outer hinge reaches support after $r_1 = 0$
t_d	rectangular pulse duration
t_f	duration of motion
w	plate deflection
M_r	radial bending moment
M_θ	circumferential bending moment
M_p	fully plastic moment
Q	shear force
R	plate radius
V	plate central velocity
$V_d, V_a, V_b, V_{ab}, V_{ba}$	plate central velocities at times $t_d, t_a, t_b, t_{ab}, t_{ba}$
W	plate central deflection
$W_d, W_a, W_b, W_{ab}, W_{ba}$	plate central deflections at times $t_d, t_a, t_b, t_{ab}, t_{ba}$
W_f	plate final central deflection
$\alpha = a/R$	loaded area parameter
$\tilde{\alpha} = 1/4^{13}$	mechanism partitioning value of α
κ_r	radial curvature
κ_θ	circumferential curvature
$\lambda = p_d/p_s$	pressure parameter
$\lambda_{1A}, \lambda_{1B}, \lambda_{2A}, \lambda_{2B}$	mechanism range functions of α
$\rho = r/R$	dimensionless radial coordinate
$\rho_0 = r_0/R$	radius of inner hinge circle
$\bar{\rho}_0 = \bar{r}_0/R$	initial radius of inner hinge circle
$\rho_1 = r_1/R$	radius of outer hinge circle
$\bar{\rho}_1 = \bar{r}_1/R$	initial radius of outer hinge circle
$\rho_a = \rho_0(t_a)$	radius of inner hinge circle at time t_a
$\rho_b = \rho_1(t_b)$	radius of outer hinge circle at time t_b.

1. INTRODUCTION

The problem treated herein is the response of a clamped circular plate subjected to a rectangular pressure pulse uniformly distributed over a central circular area, as shown in Fig. 1. The pulse is assumed to produce plastic deformations large enough to neglect elastic defor-

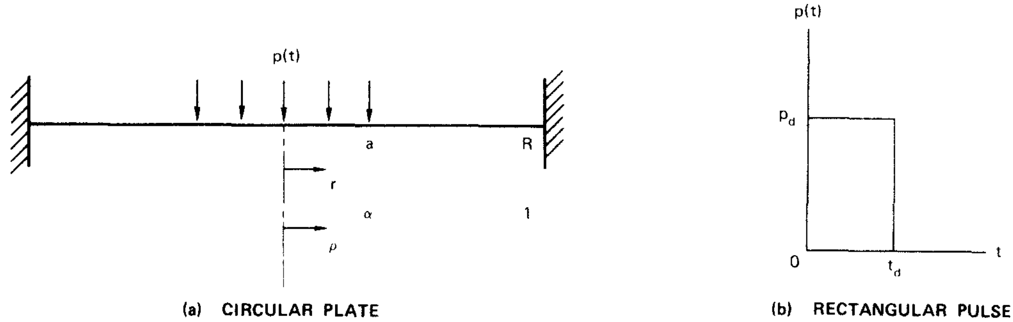

Fig. 1. Circular plate problem.

mations but small enough to neglect plastic membrane resistance. The plate material of interest is reinforced concrete so yielding is assumed to be governed by the Johansen criterion and the associated flow rule (Fig. 2). For simplicity, the analytical treatment is restricted to isotropic slabs with top and bottom steel reinforcement arranged to provide the same yield moment magnitude for positive and negative curvature changes.

A consequence of the assumed rigid-perfectly plastic behavior is that the deformation modes may be considered as simple mechanisms governed, in general, by two yield or hinge circles. Moreover, the hinge circles are stationary while the constant pressure is being applied. When the pressure is removed, the inner hinge circle moves toward the plate center while the other hinge circle moves toward the support. The final phase of deformation occurs in the static collapse mode.

The principal results of the analysis are explicit simple formulas for permanent central deflection in terms of pressure, pulse duration, loaded area radius, and plate properties (radius, areal density, yield moment). Convenient dimensionless forms of these formulas are given in terms of only two parameters, a pressure parameter $\lambda = p_d/p_s$ and an area parameter $\alpha = a/R$. In these parametric expressions, p_d is the pressure applied uniformly over a central circular area of radius a, and p_s is the static collapse pressure acting on the same circular area of a clamped circular plate of radius R.

Hopkins[1] has presented central deflection and motion duration results for simply supported and clamped circular plates subjected to impulsive loading uniformly distributed over the entire area, and has compared results for Johansen and Tresca yield criteria. Cox and Morland[2] have presented an analysis of a simply supported square plate subjected to a rectangular pressure pulse uniformly distributed over the entire area, employing the Johansen yield criterion and associated flow rule.

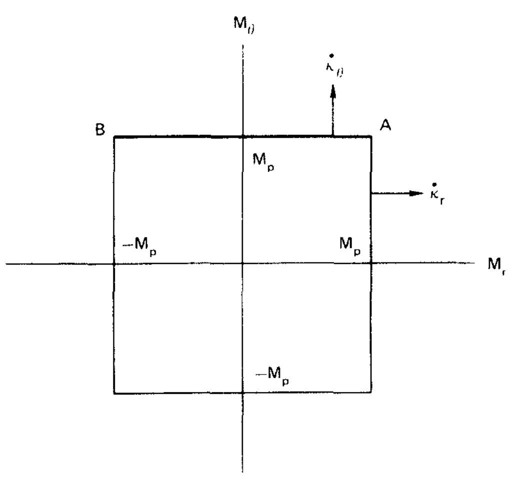

Fig. 2. Johansen's Yield criterion.

The problem treated here is similar to that of Ref. [3], where the response of a metal plate obeying the Tresca yield criterion and associated flow rule was obtained. Similar deformation mechanisms occur here but with simpler velocity fields and moment distributions because the mechanisms are associated with only one side of the Johansen yield square (AB in Fig. 2), whereas in Ref. [3] the mechanisms are associated with two sides of the Tresca yield hexagon.

2. DEFORMATION MECHANISMS

Figure 3 shows the postulated mechanisms that form immediately after the pressure is applied, when the dimensionless radius α of the loaded circle lies in the range $\tilde{\alpha} \leq \alpha \leq 1$. Figure 4 shows the initial mechanisms for the smaller loaded areas, with α in the range $0 < \alpha \leq \tilde{\alpha}$. The dividing radius, $\tilde{\alpha} = 0.63$, and the ranges for the loading parameter, λ, which determine which of the initial mechanisms 3, 2A, 2B and 1 is excited, are determined by the analysis and are shown in Fig. 5. As indicated in Figs. 3 and 4, the mechanisms are based on the use of plastic regimes A, AB and B of Fig. 2. The flow rule associated with the line AB requires no change of the radial component of curvature. Consequently, the velocity field has the spatially linear form

$$\dot{w}(r, t) = f(t)r + g(t) \tag{1}$$

where w is the plate deflection and a dot denotes time differentiation. Hinge circles are represented by the corner regimes A and B in Fig. 2.

The deformation process can be described according to the initial mechanism that is excited.

(a) *Mechanism* 3

When sufficiently high pressures are applied, initial deformation occurs by Mechanism 3, shown in Figs. 3(a) and 4(a). The plate is partitioned into three regions by two hinge circles that are stationary while the constant pressure is acting. The central circular region within the inner hinge circle $\bar{r}_0$ moves as a rigid body while the outer annular region between the outer hinge $\bar{r}_1$

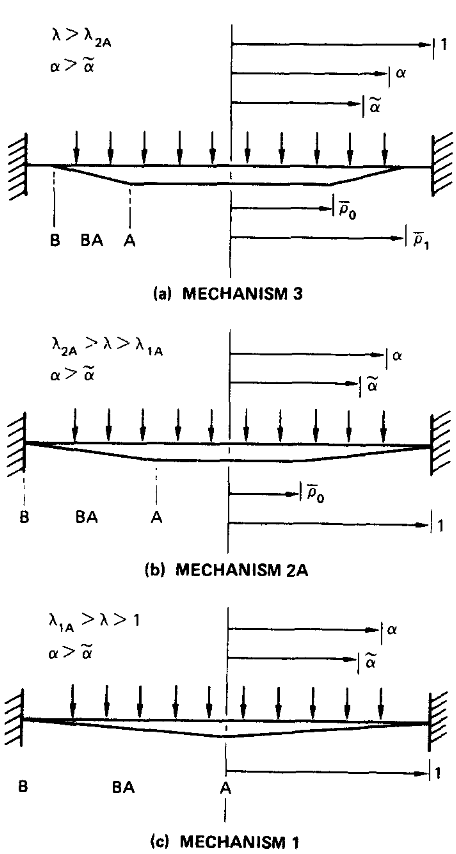

Fig. 3. Initial mechanisms ($\alpha > \tilde{\alpha}$).

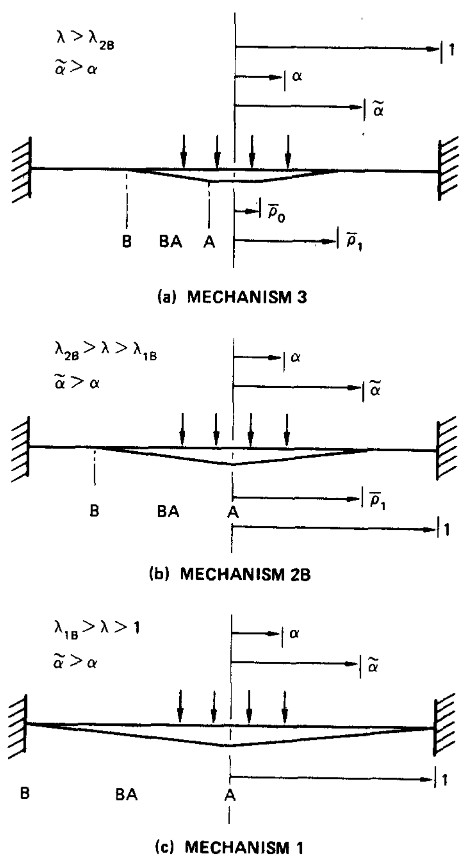

Fig. 4. Initial mechanisms ($\alpha < \tilde{\alpha}$).

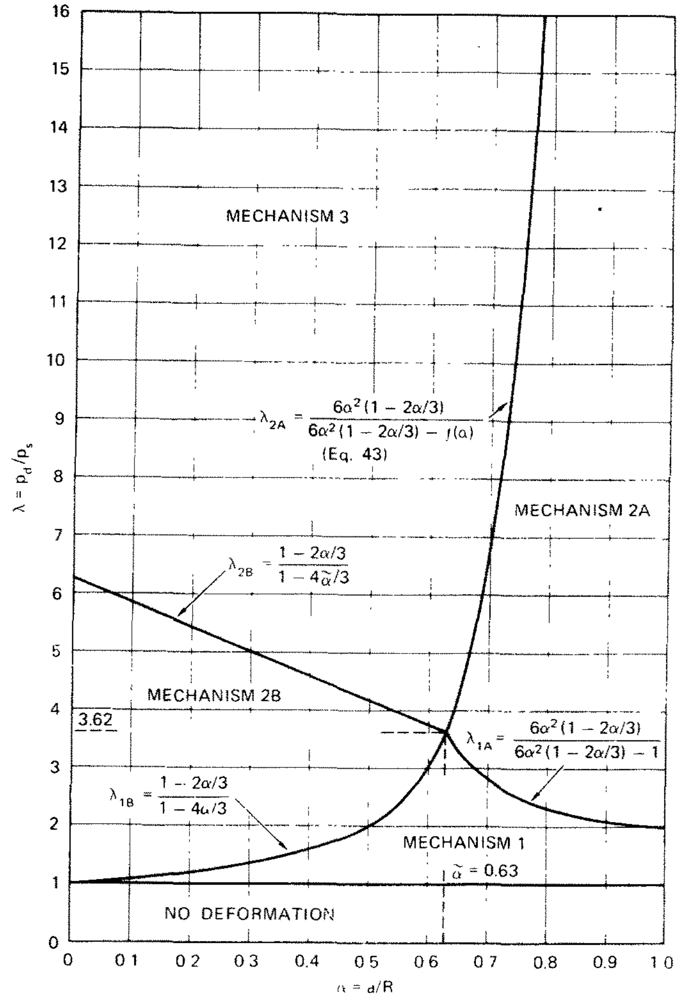

Fig. 5. Initial mechanism diagram.

and the support remains at rest. The annular region between the two hinge circles deforms into a conical shape, in accordance with the spatially linear velocity field(1). The velocity field meeting this physical description is

$$\dot{w} = \begin{cases} V & 0 \le r \le \bar{r}_0 \\ V\dfrac{\bar{r}_1 - r}{\bar{r}_1 - \bar{r}_0} & \bar{r}_0 \le r \le \bar{r}_1 \\ 0 & \bar{r}_1 \le r \le R \end{cases} \tag{2}$$

where $V = V(t)$ is the central plate velocity. At the hinge circles $\bar{r}_0$ and $\bar{r}_1$, the radial components of the bending moment are $M_r = M_p$ and $M_r = -M_p$, where M_p is the fully plastic moment per unit arc length, corresponding to plastic regimes A and B. In the region $\bar{r}_0 \le r \le \bar{r}_1$, the radial moment lies in the range $M_p \le M_r \le M_p$, corresponding to the regime AB. Also, the circumferential bending moment is $M_\theta = M_p$ throughout.

Upon removal of the pressure at time t_d (Fig. 1b), the inner hinge circle contracts to the plate center and the outer hinge circle expands to the support; the constant radii $\bar{r}_0$ and $\bar{r}_1$ in (2) are replaced by functions $r_0(t)$ and $r_1(t)$. The analysis shows (Section 3) that for the spatial loading condition $\alpha > \tilde{\alpha}$, the outer hinge circle reaches the support before the inner hinge circle reaches the center, whereupon deformation continues by Mechanism 2A [Fig. 3(b) but with a moving hinge circle $r_0(t)$]. The order of arrival is reversed for the loading condition $\alpha > \tilde{\alpha}$ and deformation continues by Mechanism 2B [Fig. 4(b) but with a moving hinge circle $r_1(t)$]. After $r_0 = 0$ and $r_1 = R$, the remaining deformation occurs by Mechanism 1 (Figs. 3c and 4c).

For the analysis, pressures are called high when they initiate deformation by Mechanism 3. High pressures are such that $\lambda > \lambda_{2A}$ for $\alpha > \tilde{\alpha}$ and $\lambda > \lambda_{2B}$ for $\alpha < \tilde{\alpha}$, where λ_{2A} and λ_{2B}, shown in Fig. 5, are determined in the analysis.

(b) *Mechanism* 2*A*

When medium pressures are applied over an area with $\alpha > \tilde{\alpha}$, initial deformation occurs by Mechanism 2A (Fig. 3b), which is a special case of Mechanism 3 because the outer hinge circle is already at the support. The velocity field is (2) with $\bar{r}_1 = R$, that is,

$$\dot{w} = \begin{cases} V & 0 \le r \le \bar{r}_0 \\ V\dfrac{R-r}{R-\bar{r}_0} & \bar{r}_0 \le r \le R. \end{cases} \tag{3}$$

Upon removal of the pressure, the hinge circle $r_0 = r_0(t)$ contracts to the plate center, and deformation is completed by Mechanism 1.

For the spatial loading condition $\alpha > \tilde{\alpha}$, pressures are called medium when they initiate deformation by Mechanism 2A. Medium pressures are such that $\lambda_{2A} > \lambda > \lambda_{1A}$ for $\alpha > \tilde{\alpha}$, where λ_{2A} and λ_{1A}, shown in Fig. 5, are determined in the analysis.

(c) *Mechanism* 2*B*

For medium pressures over an area with $\alpha < \tilde{\alpha}$, initial deformation occurs by Mechanism 2B (Fig. 4b), which is a special case of Mechanism 3 because the inner hinge is already at the center. The velocity field is (2) with $\bar{r}_0 = 0$, that is,

$$\dot{w} = \begin{cases} V\left(1-\dfrac{r}{\bar{r}_1}\right) & 0 \le r \le \bar{r}_1 \\ 0 & \bar{r}_1 \le r \le R. \end{cases} \tag{4}$$

Upon removal of the pressure, the hinge circle $r_1(t)$ expands to the support, and deformation is completed by Mechanism 1.

For the spatial loading condition $\alpha < \tilde{\alpha}$, pressures are called medium when they initiate deformation by Mechanism 2B. Medium pressures are such that $\lambda_{2B} > \lambda > \lambda_{1B}$ for $\alpha < \tilde{\alpha}$, where λ_{2B} and λ_{1B}, as shown in Fig. 5, are determined in the analysis.

(d) *Mechanism* 1

When low pressures are applied, initial deformation occurs by Mechanism 1 (Figs. 3c and 4c), which is a special case of Mechanism 3 because the inner and outer hinge circles are already at the center and support. The velocity field is (2) with $\bar{r}_0 = 0$ and $\bar{r}_1 = R$, that is,

$$\dot{w} = V\left(1-\frac{r}{R}\right) \qquad 0 \le r \le R. \tag{5}$$

Pressures are called low when they initiate deformation by Mechanism 1. Low pressures are such that $\lambda_{1A} > \lambda < 1$ for $\alpha > \tilde{\alpha}$ and $\lambda_{1B} > \lambda > 1$ for $\alpha < \tilde{\alpha}$, where λ_{1A} and λ_{1B}, shown in Fig. 5, are determined in the analysis.

For brevity, the analysis is presented in outline form and is confined to the most general case of high pressure. Results are presented only for the medium- and low-pressure cases. By considering a general pulse shape uniformly distributed over a central circular area, it is shown in the Appendix that a constant pressure produces stationary hinge circles.

3. SOLUTION FOR HIGH PRESSURES

With the aid of Fig. 6 the equations of translational and rotational motion,

$$\frac{\partial}{\partial r}(rQ) = r\left(p - m\frac{\partial^2 w}{\partial t^2}\right) \tag{6}$$

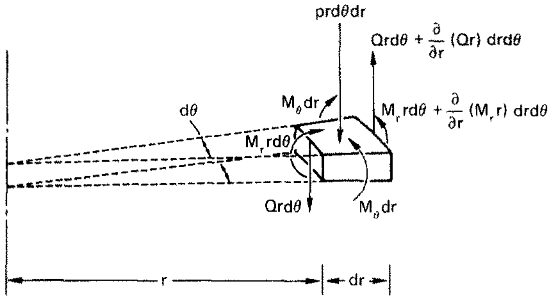

Fig. 6. Plate element with forces and moments.

and

$$\frac{\partial}{\partial r}(rM_r) - M_\theta + rQ = 0, \tag{7}$$

for a plate element can readily be derived (rotatory inertia neglected), where m is the mass per unit area. Equations (6) and (7) may be combined by eliminating the shear force Q to give

$$\frac{\partial}{\partial r}(rM_r) = M_p - \int_0^r \left(p - m\frac{\partial^2 w}{\partial t^2}\right) r'\,dr' \tag{8}$$

where the yield condition requirement $M_\theta = M_p$ throughout the plate has been used. In carrying out the spatial integrations, the moment condition, $M_p \geq M_r \geq -M_p$, between hinge circles (equality at inner and outer circles) is used; also, before the hinge circles reach their terminal positions at the center and support, $M_r = M_p$ inside the inner circle and $M_r = -M_p$ outside the outer circle.

Before solving the dynamic problem, the static collapse pressure, p_s, is obtained by integrating (8) without the inertial term. Static collapse occurs by Mechanism 1, which has the velocity field (5). Integration of (8), with the foregoing moment distribution applied to Mechanism 1, leads to

$$p_s = 4M_p/R^2\alpha^2(1 - 2\alpha/3). \tag{9}$$

(a) *Phase* 1—$p = p_d$, $0 \leq t \leq t_d$

During this constant pressure loading phase, the velocity field is given by (2) where the hinge locations $\bar{r}_0$ and $\bar{r}_1$ are constant. Substitution of (2) in (8), spatial integration, and substitution of the radial moment conditions give

$$\bar{\rho}_1{}^2 + \bar{\rho}_1\bar{\rho}_0 + \bar{\rho}_0{}^2 = 3\alpha^2[1 - (1 - 2\alpha/3)/\lambda] \tag{10}$$

and

$$\bar{\rho}_1{}^3 + \bar{\rho}_1{}^2\bar{\rho}_0 + \bar{\rho}_1\bar{\rho}_0{}^2 + \bar{\rho}_0{}^3 = 4\alpha^3 = (\alpha/\bar{\alpha})^3 \tag{11}$$

where the dimensionless radial coordinate $\rho = r/R$ has been introduced and $\bar{\alpha}$ has been simply defined by $\bar{\alpha} = 1/4^{1/3} = 0.63$. Eqns (10) and (11) determine the locations of the fixed hinge circles for given loading pressure and area parameters λ and α. The operations also give the plate central velocity and deflection, which, at time t_d, are

$$V_d = p_d t_d/m \qquad W_d = p_d t_d^2/2m. \tag{12}$$

(b) *Phase* 2—$p = 0$, $t > t_d$

Removal of the pressure sets the hinge circles in motion and the velocity becomes

$$\dot{w} = \begin{cases} V & 0 \le r \le r_0(t) \\ V\dfrac{r_1 - r}{r_1 - r_0} & r_0(t) \le r \le r_1(t). \\ 0 & r_1(t) \le r \le R \end{cases} \tag{13}$$

Integration of (8) with velocity distribution (13), and the radial moment conditions give the pair of ordinary differential equations

$$(\rho_1^2 + \rho_1\rho_0 + \rho_0^2)^{\cdot} = -3\alpha^2(1 - 2\alpha/3)/\lambda \tag{14}$$

and

$$(\rho_1^3 + \rho_1^2\rho_0 + \rho_1\rho_0^2 + \rho_0^3)^{\cdot} = 0 \tag{15}$$

for governing the motion of the hinge circles. Integration of (14) and (15) with initial conditions (10) and (11) at time t_d gives

$$\rho_1^2 + \rho_1\rho_0 + \rho_0^2 = 3\alpha^2[1 - (1 - 2\alpha/3)t/\lambda t_d] \tag{16}$$

and

$$\rho_1^3 + \rho_1^2\rho_0 + \rho_1\rho_0^2 + \rho_0^3 = 4\alpha^3 = (\alpha/\tilde{\alpha})^3. \tag{17}$$

Mechanism 3 ends when the outer hinge circle reaches the support or when the inner hinge circle reaches the center, that is, when $\rho_1 = 1$ or $\rho_0 = 0$. When $\rho_1 = 1$, eqn (17) becomes

$$1 + \rho_0 + \rho_0^2 + \rho_0^3 = (\alpha/\tilde{\alpha})^3 \tag{18}$$

and it is seen that $\rho_0 \ge 0$ only if $\alpha \ge \tilde{\alpha}$. Hence, the outer hinge circle reaches the support before the inner hinge reaches the center if the radius of the loaded area is such that $\alpha > \tilde{\alpha}$. When $\rho_0 = 0$, eqn (17) becomes

$$\rho_1 = \alpha/\tilde{\alpha}, \tag{19}$$

so $\rho_1 \le 1$ only if $\alpha \le \tilde{\alpha}$. Hence, the inner hinge circle reaches the center before the outer hinge circle reaches the support if the radius of the loaded area is such that $\alpha < \tilde{\alpha}$.

These two cases, $\alpha > \tilde{\alpha}$ and $\alpha < \tilde{\alpha}$, are treated below.

(c) *Phase 2A—$p = 0$, $t > t_a$, $\alpha > \tilde{\alpha}$*

When $\alpha > \tilde{\alpha}$, let the time of arrival of the outer hinge circle at the support be t_a, so that $\rho_1(t_a) = 1$ and $\rho_0(t_a) = \rho_a > 0$. At time t_a, eqns (16) and (17) become

$$1 + \rho_a + \rho_a^2 = 3\alpha^2[1 - (1 - 2\alpha/3)t_a/\lambda t_d] \tag{20}$$

and

$$1 + \rho_a + \rho_a^2 + \rho_a^3 = (\alpha/\tilde{\alpha})^3 \tag{21}$$

which determine the time of arrival t_a and the initial position ρ_a of the remaining mobile hinge circle for subsequent deformation by Mechanism 2A. For $t > t_a$, the velocity field is

$$\dot{w} = \begin{cases} V = V(t_a) = V(t_d) & 0 \le r \le r_0(t) \\ V\dfrac{R - r}{R - r_0} & r_0(t) \le r \le R. \end{cases} \tag{22}$$

Integration of (8) with velocity distribution (22) leads to

$$(\rho_0 + \rho_0^2 - \rho_0^3)^{\cdot} = -6\alpha^2(1 - 2\alpha/3)/\lambda t_d. \tag{23}$$

Integration of (23) with initial conditions (20) and (21) leads to

$$t_{ab}/t_d = \lambda/\lambda_{1A} \qquad \lambda_{1A} = 6\alpha^2(1 - 2\alpha/3)/[6\alpha^2(1 - 2\alpha/3) - 1] \tag{24}$$

where t_{ab} is the time at which the hinge arrives at the center, that is, $\rho_0(t_{ab}) = 0$. Because the central region of plate within the contracting hinge circle is moving at constant velocity V_d during this phase of deformation, the deflection at time t_{ab} is, by eqns (12) and (24),

$$W_{ab} = W_d + V_d(t_{ab} - t_d) = (p_d t_d^2/2m)(2\lambda/\lambda_{1A} - 1). \tag{25}$$

(d) *Phase* 3—$p = 0$, $t < t_{ab}$, $\alpha > \tilde{\alpha}$

The remaining deformation occurs by Mechanism 1, which has the velocity field of (5). Integration of (8) with velocity distribution (5) gives duration of motion t_f and the final central deflection W_f in the form

$$t_f/t_d = \lambda \tag{26}$$

and

$$W_f = W_{ab} + V_{ab}(t_f - t_{ab})/2 \tag{27}$$

where $V(t_f) = 0$ and $V_{ab} = V(t_{ab}) = V_d$. Thus, substitution of (25), (26), (24), and (12) for W_{ab}, t_f, t_{ab} and V_d in (27) gives the final central deflection

$$W_f = (p_d t_d^2/2m)[\lambda(1 + \lambda_{1A}^{-1}) - 1]. \tag{28}$$

(e) *Phase* 2*B*—$p = 0$, $t > t_b$, $\alpha < \tilde{\alpha}$

Turning now to the case where the loaded area is such that $\alpha < \tilde{\alpha}$, let the time of arrival of the inner hinge circle at the center be t_b, so that $\rho_0(t_b) = 0$ and $\rho_1(t_b) = \rho_b < 1$. Then (16) and (17) become

$$\rho_b^2 = 3\alpha^2[1 - (1 - 2\alpha/3)t_b/\lambda t_d] \tag{29}$$

and

$$\rho_b = \alpha/\tilde{\alpha} \tag{30}$$

which determine the time of arrival t_b and the initial position ρ_b of the remaining mobile hinge for subsequent deformation by Mechanism 2B. For $t > t_b$, the velocity field is

$$\dot{w} = \begin{cases} V(1 - r/r_1) & 0 \le r \le r_1(t) \\ 0 & r_1(t) \le r \le R. \end{cases} \tag{31}$$

Integration of (8) with velocity distribution (31) leads to the equations

$$\dot{V}\rho_1^2 + V\rho_1\dot{\rho}_1 = -6\alpha^2(1 - 2\alpha/3)p_d/\lambda m \tag{32}$$

and

$$\dot{V}\rho_1^2 + 2V\rho_1\dot{\rho}_1 = -3\alpha^2(1 - 2\alpha/3)p_d/\lambda m \tag{33}$$

governing the central plate velocity and the hinge location. The solutions of (32) and (33) satisfying Phase 2B initial conditions (29) and (30) are

$$V = \left[\frac{1-(1-2\alpha/3)t/\lambda t_d}{1-(1-2\alpha/3)t_b/\lambda t_d}\right]^3 V_b$$

and

$$\rho_1 = \left[\frac{1-(1-2\alpha/3)t_b/\lambda t_d}{1-(1-2\alpha/3)t/\lambda t_d}\right]\rho_b. \tag{34}$$

In (34), $V_b = V_d = p_d t_d/m$ and, from (29) and (30),

$$t_b/t_d = \lambda/\lambda_{2B} \qquad \lambda_{2B} = (1-2\alpha/3)/(1-4\bar{\alpha}/3). \tag{35}$$

The time of arrival t_{ba} of the hinge at the support, determined by setting $\rho_1(t_{ba}) = 1$ in (34), is given by

$$t_{ba}/t_d = \lambda/\lambda_{1B} \qquad \lambda_{1B} = (1-2\alpha/3)(1-4\alpha/3) \tag{36}$$

and the central plate velocity and deflection at this time are

$$V_{ba} = 4\alpha^3 p_d t_d/m \tag{37}$$

and

$$\begin{aligned} W_{ba} &= W_d + V_d(t_b - t_d) + 4V_d t_d(\bar{\alpha}^4 - \alpha^4)\lambda/3(1-2\alpha/3) \\ &= (p_d t_d^2/2m)(2\lambda/\lambda_{1B} - 1). \end{aligned} \tag{38}$$

(f) *Phase* 3—$p = 0$, $t > t_{ba}$, $\alpha < \bar{\alpha}$

The remaining deformation occurs by Mechanism 1, which has the velocity field of (5) as in the loading case $\alpha > \bar{\alpha}$ but with initial velocity V_{ba} of (37) and initial deflection W_{ba} of (38). The duration of motion and final central deflection are

$$t_f/t_d = \lambda \tag{39}$$

and

$$\begin{aligned} W_f &= (p_d t_d^2/2m)[2(1-\bar{\alpha})\lambda/(1-2\alpha/3) - 1] \\ &= (p_d t_d^2/2m)[\lambda(2 + 3\lambda_{2B}^{-1} - \lambda_{1B}^{-1})/2 - 1]. \end{aligned} \tag{40}$$

4. PRESSURE RANGES

(a) *Mechanism* 3

When Mechanism 3 is excited by a rectangular pulse, the stationary hinge locations are determined by eqns (10) and (11). A smallest pressure exciting Mechanism 3 is obtained by considering λ, α pairs that produce an outer hinge circle at the support while still producing an inner hinge circle, that is, $\bar{\rho}_0 > 0$ and $\bar{\rho}_1 = 1$. Equations (10) and (11) then become

$$1 + \bar{\rho}_0 + \bar{\rho}_0^2 = 3\alpha^2[1-(1-2\alpha/3)/\lambda] \tag{41}$$

and

$$1 + \bar{\rho}_0 + \bar{\rho}_0^2 + \bar{\rho}_0^3 = (\alpha/\bar{\alpha})^3. \tag{42}$$

Equation (42) shows that the smallest pressure cases can arise only if $\alpha > \bar{\alpha}$, to ensure that

$\bar{\rho}_0 > 1$. Solving for λ gives the required pressure parameter

$$\lambda_{2A} = 6\alpha^2(1 - 2\alpha/3)/[6\alpha^2(1 - 2\alpha/3) - (1 + \bar{\rho}_0 + \bar{\rho}_0^2 - \bar{\rho}_0^3)] \tag{43}$$

where $\bar{\rho}_0$ is the solution of (42).

Another smallest pressure stimulating Mechanism 3 is obtained by considering λ, α pairs that produce the limiting inner hinge circle at the center while still producing an outer hinge circle, that is, $\bar{\rho}_0 = 0$ and $\bar{\rho}_1 < 1$. Equations (10) and (11) then become

$$\bar{\rho}_1^2 = 3\alpha^2[1 - (1 - 2\alpha/3)/\lambda] \tag{44}$$

and

$$\bar{\rho}_1 = \alpha/\tilde{\alpha}. \tag{45}$$

Equation (45) shows that the smallest pressure cases can arise only if $\alpha < \tilde{\alpha}$. Solving for λ gives the required pressure parameter

$$\lambda_{2B} = (1 - 2\alpha/3)/(1 - 4\tilde{\alpha}/3). \tag{46}$$

(b) *Mechanism* 2*A*

When Mechanism 2A is excited by a rectangular pulse analysis similar to that outlined in Section 3, shows that the location $\bar{\rho}_0$ of the stationary hinge circle is the solution of

$$1 + \bar{\rho}_0 + \bar{\rho}_0^2 - \bar{\rho}_0^3 = 6\alpha^2(1 - 2\alpha/3)(1 - \lambda^{-1}). \tag{47}$$

The smallest pressure stimulating Mechanism 2A is therefore given by (47) with $\bar{\rho}_0 = 0$, that is,

$$\lambda_{1A} = 6\alpha^2(1 - 2\alpha/3)/[6\alpha^2(1 - 2\alpha/3) - 1]. \tag{48}$$

(c) *Mechanism* 2*B*

When Mechanism 2B is excited, analysis shows that the stationary hinge circle location $\bar{\rho}_1$ is given by

$$4\alpha/3\bar{\rho}_1 = 1 - (1 - 2\alpha/3)/\lambda. \tag{49}$$

The smallest pressure is found by setting $\bar{\rho}_1 = 1$ in (49) to give

$$\lambda_{1B} = (1 - 2\alpha/3)/(1 - 4\alpha/3). \tag{50}$$

(d) *Mechanism* 1

Mechanism 1 is excited by pressures in the range $1 < \lambda < \lambda_{1A}$ when $\alpha > \tilde{\alpha}$ and the range $1 < \lambda < \lambda_{1B}$ when $\alpha < \tilde{\alpha}$.

The pressure parameters λ_{2A}, λ_{2B}, λ_{1A} and λ_{1B} given by (43), (46), (48) and (50) as functions of α are shown in Fig. 5.

5. DEFLECTION RESULTS FOR MEDIUM AND LOW PRESSURES

By performing analyses like that outlined in Section 3 for high pressures, the following central deflection formulas can be derived for medium and low pressures.

Mechanism 2A: $\lambda_{1A} \leq \lambda \leq \lambda_{2A} \quad \tilde{\alpha} \leq \alpha \leq 1$

$$W_f = (p_d t_d^2/2m)[\lambda(1 + \lambda_{1A}^{-1}) - 1]. \tag{51}$$

Mechanism 2B: $\lambda_{1B} \leq \lambda \leq \lambda_{2B} \quad 0 \leq \alpha \leq \tilde{\alpha}$

$$W_f = (p_d t_f^2/2m)2(\alpha/\bar{\rho}_1)^3[\lambda(2 - \lambda_{1B}^{-1}) + 1] \tag{52}$$

where

$$\alpha/\bar{\rho}_1 = (3/4)[1-(1-2\alpha/3)/\lambda] = (3/4)[1-\lambda_{1B}/\lambda(2\lambda_{1B}-1)].$$

Mechanism 1: $1 \le \lambda \le \lambda_{1A}$ $\bar{\alpha} \le \alpha \le 1$

$1 \le \lambda \le \lambda_{1B}$ $0 \le \alpha \le \bar{\alpha}$

$$\begin{aligned} W_F &= (p_d t_d^2/2m)6\alpha^2(1-2\alpha/3)(\lambda-1) \\ &= (p_d t_d^2/2m)\lambda_{1A}(\lambda-1)/(\lambda_{1A}-1) \\ &= (p_d t_d^2/2m)27(\lambda_{1B}-1)^2\lambda_{1B}(\lambda-1)/2(2\lambda_{1B}-1)^3. \end{aligned} \tag{53}$$

In all cases the duration of motion is given by

$$t_f/t_d = \lambda.$$

Figure 7 shows several central deflection curves obtained from formulas (28), (40), (51), (52) and (53). The deflection is made dimensionless by dividing it by the quantitity $p_d t_d^2/2m$, the free flight deflection of that part of the plate under the loading at the pulse duration time t_d. The static collapse pressure p_s used in the loading parameter expression $\lambda = p_d/p_s$ is given by (9).

6. CONCLUSIONS

The results are potentially useful as a simple design aid for reinforced concrete slabs subjected to pulse loading. However, to establish confidence, the deflection predictions require comparison with experimental results and finite-element code predictions.

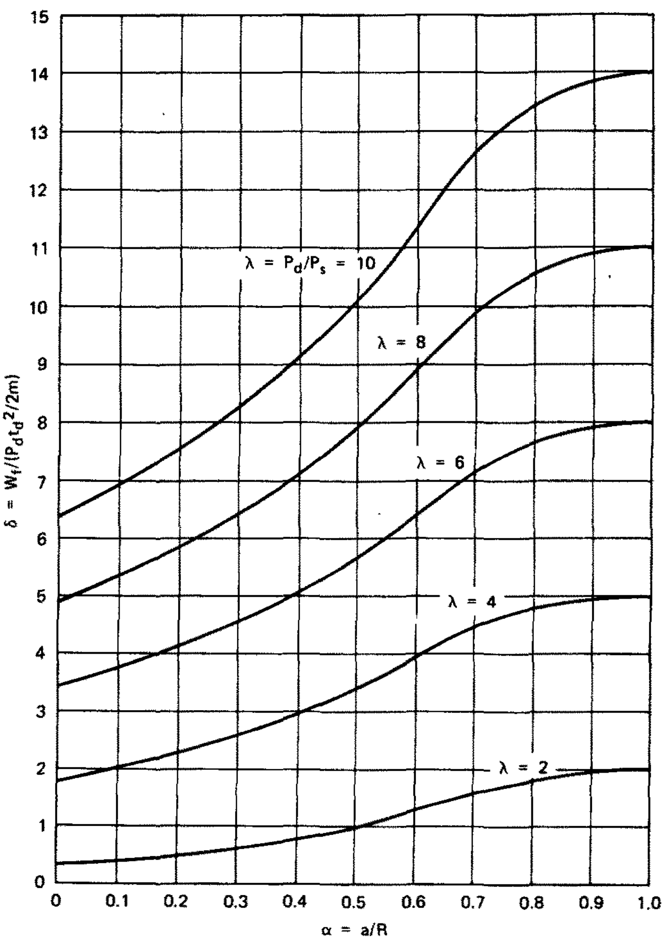

Fig. 7. Central deflection curves.

The basic assumption of rigid-plastic behavior in a predominantly bending response inherently imposes a range of applicability. The importance of axial restraints in beams and radial restraints in plates for causing a build-up of membrane forces has been stressed by Symonds and Mentel [4], Woods [5], Jones [6] and others [7]. As an approximate guide, membrane forces are probably important when the deflections exceed the slap thickness.

REFERENCES

1. H. G. Hopkins, Some remarks concerning the dependence of the solution of plastic plate problems upon the yield condition. *Proc. 9th Inter. Cong. Appl. Mech.* Vol. 6, pp. 448–457, Brussels (1957).
2. A. D. Cox and L. W. Morland, Dynamic plastic deformations of simply-supported square plates. *J. Mech. and Physics of Solids* **7**, 229–241 (1959).
3. A. L. Florence, Clamped circular rigid-plastic plates under central blast loading. *Int. J. Solids Structures* **2**, 319–335 (1966).
4. P. S. Symonds and T. J. Mentel, Implusive loading of plastic beams with axial restraints. *J. Mech. and Physics of solids* **6**, 186–202 (1958).
5. R. H. Woods, *Plastic and elastic design of slabs and plates.* The Ronald Press Company, New York (1961).
6. N. Jones, Impulsive loading of a simply supported circular rigid plastic plate. *J. Appl. Mech.* **35**, 59–65 (1968).
7. A. L. Florence, Circular plate under a uniformly distributed impulse. *Int. J. Solids Structures* **2**, 37–47 (1966).

APPENDIX

This appendix examines the assertion that the hinge circles are stationary under a constant pressure. Mechanism 3 with moving hinge circles has the velocity field of (13). If this velocity field and a time-varying pressure is substituted in the equation of motion (8) and spatial integration is performed, the following two differential equations are obtained

$$I\dot{y} = p(4a^3 - y) \tag{A1}$$

$$I\dot{z} = p(4a^2 - z) - 12M_p \tag{A2}$$

where

$$y = r_1^3 + r_1^2 r_0 + r_1 r_0^2 + r_0^3$$

$$z = r_1^2 + r_1 r_0 + r_0^2$$

$$I = \int_0^t p(\tau)\,\mathrm{d}\tau.$$

If the initial pressure is $p(o) = \bar{p}$ and the initial hinge circle locations are $\bar{r}_0$ and $\bar{r}_1$, giving initial valyes $\bar{y}$ and $\bar{z}$, it follows that

$$\bar{y} = 4a^3 \qquad \bar{z} = 3a^2 - 12M_p/\bar{p}. \tag{A3}$$

Examination of eqn (A1) with the initial values of (A3), by Taylor series expansions about the time origin, for example, reveals that

(1) For $p = p(t)$, the solutions are $y = \bar{y}$ and $z = z(t)$
(2) For $p = \bar{p}$, a constant, the solutions are $y = \bar{y}$ and $z = \bar{z}$.

For a constant pressure $p = p_d$, the results $\dot{y} = 0$ and $\dot{z} = 0$ mean that

$$(3r_1^2 + 2r_1 r_0 + r_0^2)\dot{r}_1 + (r_1^2 + 2r_1 r_0 + 3r_0^2)\dot{r}_0 = 0$$

$$(2r_1 + r_0)\dot{r}_1 + (r_r + 2r_0)\dot{r}_0 = 0.$$

The determinant of the coefficients of the hinge circle velocities is

$$\Delta = (r_1 - r_0)(r_1^2 + 4r_1 r_0 + r_0^2)$$

and, for positive r_1 and r_0, $\Delta = 0$ only if $r_1 = r_0$. Only for an ideal impulse (unit of infinite pressure times zero duration) is $r_1 = r_0$. Hence, for a constant applied pressure, $\Delta \neq 0$ and the hinge circles are stationary.